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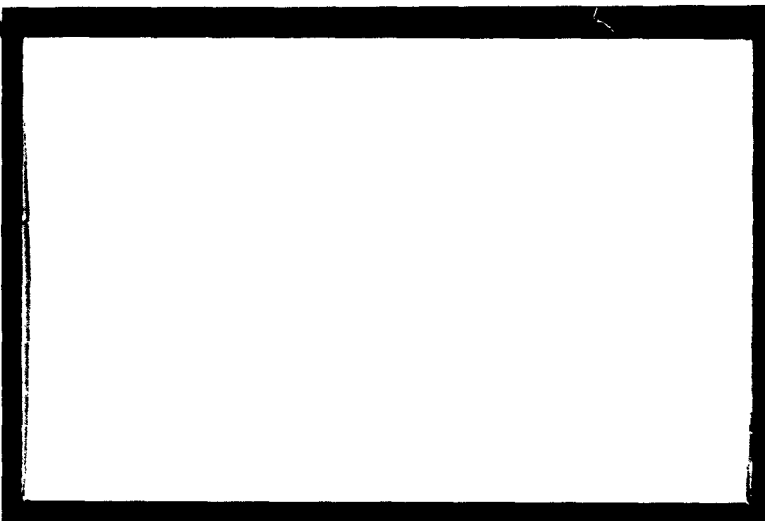
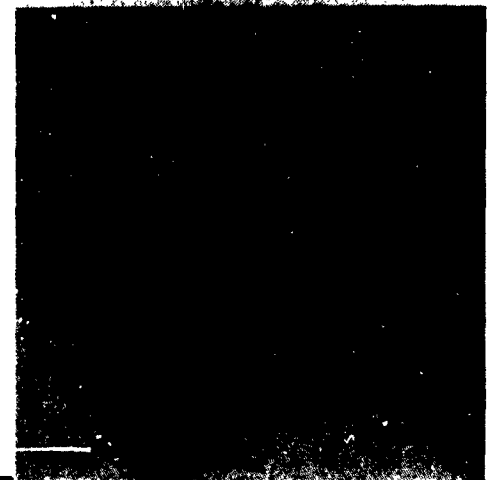
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**ON THE MOTION OF A LIQUID IN A  
SPHEROIDAL CAVITY OF A PRECESSING  
RIGID BODY**

**P. H. Roberts and K. Stewartson**

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## ABSTRACT

Neglecting magnetic effects the precession of the mantle of the Earth is responsible for hydrodynamic motions in the liquid core because, (i) the velocity of the inner boundary of the mantle along its normal is non-zero unless the core boundary is a perfect sphere, (ii) there is a viscous coupling between core and mantle, (iii) there is a magnetic coupling between core and mantle because lines of geomagnetic force thread each. In this preliminary discussion of the problem, we suppose that the core boundary is an oblate spheroid of eccentricity  $e$ ; it is shown that the relative importance of (i) and (ii) depends in an essential way on the value of  $R_1 = \omega e^2 / \Omega$  where  $\omega$  is the angular velocity of rotation and  $\Omega (\ll \omega)$  is the angular velocity of precession. Since  $R_1$  is almost certainly large for the Earth, the present discussion is limited to this case.

The motion of the fluid is found by considering an initial value problem in which the axis of rotation of the spheroid is impulsively moved at time  $t = 0$ ; before that time this axis is supposed to be fixed in space, the fluid and envelope turning about it as a solid body. The solution is divided into a steady motion and transients, and, by evaluating the effects of the viscous boundary layer, the transients are shown to decay with time. The steady motion which remains is a circulation with constant vorticity in planes perpendicular to  $\omega$  and  $\Omega$ , the streamlines being similar and similarly situated ellipses.

ON THE MOTION OF A LIQUID IN A SPHEROIDAL  
CAVITY OF A PRECESSING RIGID BODY

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1. INTRODUCTION

Some years ago, Bondi and Lyttleton (1) examined the motion of a liquid in a spherical cavity of a rigid body which was rotating with an angular velocity  $\omega$  about an axis through the centre of the sphere when this axis, in turn, was precessing with angular velocity  $\Omega$  ( $\Omega \ll \omega$ ) about an axis fixed in space. The aim of their investigation was to throw some light on the effect of the precession of the Earth on the motion of the liquid core. The analytical approach used by Bondi and Lyttleton was to suppose that the fluid is Newtonian and practically inviscid. Then, in the first approximation, viscosity was neglected, and it was hoped to correct for viscosity by a thin boundary layer near the surface of the cavity. Unfortunately, on attempting to solve the steady inviscid equations, they arrived at a definite contradiction

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which strongly suggested to them that no steady state motion of a permanent character is possible for the fluid.

The spherical cavity presents a special difficulty in the theory of rotating fluids because without viscosity no motion can be communicated to the fluid by a rotation of the boundary. The angular velocity acquired by the fluid, even if the sphere is not precessing, is due entirely to the generation of vorticity in the boundary layer at the surface of the cavity and one might expect it to take a time  $O(a^2/\nu)$ , where  $a$  is the radius of the cavity and  $\nu$  the kinematic viscosity, for the motion of the fluid and sphere to rotate together as a rigid body. Let us suppose this time has passed, that the precessing motion is set up at  $t = 0$ , and that subsequently the fluid may be regarded as inviscid. Then while the solid body precesses the fluid rotates about a fixed axis, being unaffected by the motion of its envelope. The motion of the fluid is actually steady but relative to the envelope it appears to be unsteady, its axis of rotation precessing with angular velocity  $-\Omega$  and, when  $\Omega t$  is small, this means that the velocity perturbations are proportional to  $\Omega t$ . Thus the contradiction obtained by Bondi and Lyttleton does not necessarily mean that the motion of the fluid is not ultimately steady. For a real fluid, one might expect that the motion described above would occur if  $\Omega a^2/\nu \gg 1$ , and if, on the other hand,  $\Omega a^2/\nu \ll 1$ , the motion of the fluid and envelope would be virtually a rigid body precession.

This discussion strongly supports their view that the determination of the motion is likely to prove a problem of great difficulty, and consequently we thought it desirable to begin our attack by studying the related problem of the

oblate spheroidal envelope which includes the sphere as a special case but proves easier because the motion of the boundary of the cavity must communicate itself in part to the fluid. This problem is not without interest in the geophysical context with which Bondi and Lyttleton were concerned because, although one can say that the core of the Earth is nearly a sphere in that

$$\frac{a^2 - b^2}{a^2} \ll 1 ,$$

where  $a$  and  $b$  are the semi-major and minor axes of the core, one cannot be sure of the values of

$$R_1 = \frac{(a^2 - b^2)\omega}{\Omega a^2} , \quad R_2 = \frac{\Omega a^2}{\nu} ,$$

which, with  $R_3 = \omega a^2 / \nu ( \gg 1 )$ , control the motion of the fluid in the core relative to the rigid body rotation. If  $R_1 = 0$ , the motion is critically dependent on  $R_2$  while if  $R_1$  is large the motion is independent of  $R_2$  except, possibly, if  $R_1 R_2 \lesssim 1$ . Our investigation here is based on the assumptions that  $R_1 \gg 1$ ,  $R_1 R_2 \gg 1$ ; it is hoped to explore other limiting situations in a later paper. These assumptions are not unreasonable in the geophysical context since, for the surface of the Earth,  $(a^2 - b^2)/a^2 \approx 1/300$ , and the value of this parameter for the core is not likely to be an order of magnitude less. Further,  $\omega/\Omega \approx 5.10^6$ , and  $\nu$  is almost certainly much less than  $10^8 \text{ cm}^2/\text{sec}$ . However, it should not be forgotten that our results

are extremely sensitive to  $(a^2 - b^2)/a^2$  when  $a \approx b$ . It may also be appropriate to remark that, as a geophysical model, our present approach involves a gross oversimplification of the dynamics of the core since it excludes magnetic forces, and these are likely to be the most potent agency in coupling core and mantle (cf., e.g., Bullard, 2). Of course, a realistic model including magnetic forces would be immensely complicated (although, in its favour, it should be mentioned that the case  $a = b$  would no longer be singular!). Nevertheless, we believe our simple model has some bearing on the generation of the geomagnetic field. It has long been known (2) that the only processes which are energetically capable of generating the geomagnetic field by dynamo action are thermal convection and precession. Bullard (2), while himself favouring the former of these mechanisms, was unable to rule out the latter. Indeed, the subsequent suggestion (1) that turbulence might be generated by the precessional effects near the singularities of the boundary layer at latitudes  $30^\circ\text{N}$  and  $30^\circ\text{S}$  made it seem (3) not unlikely that precessional motion played an important part in the geomagnetic dynamo. However, turbulence was invoked by Bondi and Lyttleton as a motion which the fluid would be likely to adopt in the absence of a permanent steady-state flow. In our model, on the other hand, a steady state flow does exist, and turbulence could arise only if it is unstable (see below). If we exclude this possibility, it is clear that, even though in our model the order of magnitude of the precessional motions ( $\sim 1 \text{ cm./sec.}$  near the core boundary) should suffice to make their inductive effects felt, their structure is far too simple, in regard to the restrictions imposed by Cowling's theorem (4), to maintain dynamo action. Other motions would be

required to complete the regenerative cycle. In principle, even a simple differential rotation about the axis of  $\omega$  might suffice (5), and it may be worth noting that such motions might arise if  $R_1 \ll 1$  in the core.

Even for an oblate spheroidal cavity containing almost inviscid fluid the determination of the flow has some unusual features. Thus the governing equations for an inviscid fluid are hyperbolic while the boundary condition is a relation between the function and its normal derivative. Such a boundary condition is appropriate to an elliptic differential equation and there are no corresponding existence and uniqueness theorems for hyperbolic equations. In fact one can construct examples of non-existence and non-uniqueness as follows.

Suppose that the governing equation is

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} = 0 \quad (1.1)$$

and that (a)  $\phi = 0$  on the circle  $x^2 + y^2 = 1$ , or (b)  $\phi = 0$  on three sides of the square  $|x| \leq 1$ ,  $|y| \leq 1$  and  $\phi = 1$  on the fourth side. One solution of (a) is

$$\phi = A(x^2 + y^2 - 1)$$

inside the circle, where  $A$  is arbitrary. This solution can be added to any other, so that there is no unique solution of (1.1) in this case. Again every solution of (1.1) which vanishes on three sides of a square must vanish on



the fourth side so that the solution of problem (b) cannot exist.

Fortunately in the present problem there is a simple physical explanation of these phenomena. The non-uniqueness is associated with the existence of free oscillatory motions of the fluid in which the boundary remains fixed and the non-existence is associated with the resonance which can occur when the boundary is made to oscillate with the same period as that of one of the free oscillations of the fluid.

The difficulty about the inappropriate type of boundary condition is removed by considering motions which start from a position of relative rest. We suppose that the fluid and envelope are initially rotating as if solid with angular velocity  $\omega$  and that at time  $t = 0$  the envelope is set precessing with angular velocity  $\Omega$ . The equations governing the induced motion of the fluid are unsteady but the time dependence is removed on applying a Laplace transformation with parameter  $s$  and the equation is reduced to a single one of Laplace's form apart from a scaling factor which is a function of  $s$ . A general solution can be written down and, on inversion, the history of the motion can be traced. From an examination of the solution at large times we can relate the non-uniqueness to free oscillations set up by the initial motion and the non-existence to resonance.

For a spheroid the actual inviscid solution is straightforward being effectively given by Hough and by Poincaré (6), the difficulty found by Br. 11 and Lyttleton for  $a = b$  is seen to be a resonance and explicable on the ... stated in physical terms at the beginning of this section.

In §4 the boundary layer due to this motion is investigated. The inviscid motion outside consists of two parts (i) dependent on the azimuth angle  $\theta$  and due to the steady motion of the boundary, (ii) independent of  $\theta$  and due to the angular velocity which the boundary acquired initially but failed to communicate immediately to the fluid. The boundary layer associated with (i) was studied by Bondi and Lyttleton but we repeat their investigation here since we are including time as a parameter. It develops a singularity on certain circles of latitude as  $s$  approaches any value on the imaginary axis between  $-3i$  and  $i$ . The boundary layer associated with (ii) has also been studied earlier by Stewartson (7) and Proudman (8), and it develops singularities as  $s$  approaches any value on the imaginary axis between  $\pm 2i$ .

Bondi and Lyttleton (1) noticed that the normal velocity on the critical circle becomes infinite in the limit and suggested that the flow near it was unstable leading to turbulence here and elsewhere. Although one cannot be sure without carrying out an actual experiment, more recent investigations do not support this view. The singularity on a critical line is a singularity in the boundary layer sense, i. e., if  $v_n$  is the normal velocity just outside the boundary layer, the equations imply that  $v_n = O(\nu^{\frac{1}{2}})$  almost everywhere and on approaching the critical line  $\nu^{-\frac{1}{2}} v_n \rightarrow \infty$ . This does not necessarily mean that  $v_n \rightarrow \infty$  and indeed, in a related problem (9), a more detailed investigation of the flow properties near the critical line implied  $v_n = O(\nu^{\frac{1}{2}})$  which is still small in the limit  $\nu \rightarrow 0$ . Further a detailed solution of a related problem to (ii) by Stewartson (10) elucidates the role of the

critical circles as the origin of shear layers which penetrate into the fluid, transporting fluid from one part of the boundary to another via the interior and also adjusting the angular velocity of the main body of fluid when required. Free shear layers are certainly notoriously prone to instability; however, after Proudman (8) had postulated their existence in a discussion of the flow between two concentric spheres, experiments carried out to test his conclusions did not, in all cases, reveal instability. It was found (11) that, if the angular velocity of the inner sphere exceeded that of the outer sphere, the shear layer was rapidly destroyed by Taylor-Görtle instabilities and by the formation of Kármán vortex sheets at the layer. If however, the outer sphere rotated more rapidly than the inner sphere, the shear layer showed scarcely any tendency to instability.

In §5 and §6 the (tertiary) modification to the secondary flow induced by these boundary layers is discussed with particular reference to the oscillations produced by the initial motion of the envelope. It is shown that, of the motions dependent on  $\theta$ , only the steady motion survives as  $t \rightarrow \infty$ . The effect of the boundary layer independent of  $\theta$  is to lead to a breakdown in the tertiary flow as  $t \rightarrow \infty$ , and it is argued that this must mean that, relative to the boundary, the motion independent of  $\theta$  must die out as  $t \rightarrow \infty$  due to the communication of vorticity to the fluid via the boundary layer.

## 2. EQUATIONS OF UNSTEADY MOTION

We consider a mass of incompressible fluid which occupies the whole of a rigid envelope whose internal boundary,  $S_0$ , is a surface of revolution with axis of symmetry  $L_B$ . Initially the fluid and the envelope are rotating about the axis with angular velocity  $\omega$  which is an absolute constant. At time  $t = 0$ , the axis  $L_B$  is set rotating with a small uniform angular velocity  $\Omega$  about an axis  $L_S$  fixed in space which intersects  $L_B$  at a point  $O$ , and which is inclined at an angle  $\alpha$  to it. For definiteness, we shall suppose that the perturbed motion is started impulsively so that  $\Omega$  is also an absolute constant. We wish to find the subsequent motion of the fluid.

Consider a reference frame,  $\mathcal{F}$ , rotating with angular velocity  $\Omega$  relative to which  $L_B$  and  $L_S$  are stationary, and let the velocity of the fluid relative to  $\mathcal{F}$  be

$$\mathcal{V} = \mathcal{U} + \omega \times \mathcal{L} \quad (2.1)$$

At  $t = 0^-$ ,  $\mathcal{U} = 0$ , and at  $t = 0^+$ ,  $\mathcal{U} \neq 0$  in virtue of the impulsive motion of  $S_0$ , but it will be determinate from the initial motion of  $S_0$ . The actual velocity,  $\mathcal{V}$ , of the fluid, relative to axes fixed in space and instantaneously coinciding with  $\mathcal{F}$ , is

$$\mathcal{V} = \mathcal{U} + \omega \times \mathcal{L} + \Omega \times \mathcal{L} \quad (2.2)$$

We know that, at  $t = 0+$ ,  $\mathbf{u} + \mathbf{Q} \times \mathbf{r}$  is irrotational and uniquely determinate from the motion of  $S_0$ , and hence is also known at  $t = 0+$ . The equations of motion relative to are

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{Q} \times \mathbf{v} + (\mathbf{v} \cdot \text{grad}) \mathbf{v} = -\text{grad} \left( W + \frac{p}{\rho} \right) + \nu \nabla^2 \mathbf{v}, \quad (2.3)$$

$$\text{div } \mathbf{v} = 0, \quad (2.4)$$

where  $p$  is the pressure,  $\rho$  the density,  $\nu$  the kinematic viscosity, and  $W$  the potential per unit mass of the external forces, supposed conservative, which act on the fluid. We now express (2.3) in terms of  $\mathbf{u}$ , obtaining

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + 2(\mathbf{u} + \mathbf{Q}) \times \mathbf{u} - (\mathbf{u} + \mathbf{u} \times \mathbf{r}) \times \text{curl } \mathbf{u} \\ = -\text{grad} \left[ W + \frac{p}{\rho} - \frac{1}{2} \{ (\mathbf{u} + \mathbf{Q}) \times \mathbf{r} \}^2 \right] + \nu \nabla^2 \mathbf{u} + \mathbf{r} \times (\mathbf{Q} \times \mathbf{u}) \\ + \mathbf{u} \cdot (\mathbf{u} \times \mathbf{r}) + \frac{1}{2} \mathbf{u}^2 \end{aligned}$$

whence, on neglecting squares and products of  $\mathbf{u}$  and  $\mathbf{Q}$ , we find

$$\frac{\partial \mathbf{u}}{\partial t} + 2\mathbf{u} \times \mathbf{u} - (\mathbf{u} \times \mathbf{r}) \times \text{curl } \mathbf{u} = -\omega \text{ grad } V' + \nu \nabla^2 \mathbf{u} - 2(\mathbf{Q} \cdot \mathbf{r}) \mathbf{u}, \quad (2.5)$$

where

$$\omega V' = W + \frac{p}{\rho} + \frac{1}{2} \mathbf{v}^2 - (\mathbf{u} \times \mathbf{r})^2 + (\mathbf{u} \cdot \mathbf{r})(\mathbf{Q} \cdot \mathbf{r}) - 2(\mathbf{Q} \cdot \mathbf{u})r^2. \quad (2.6)$$

This equation differs from that given by Bondi and Lyttleton (1), viz.

$$\left[ \frac{\partial \mathbf{u}}{\partial t} \right] + 2\boldsymbol{\omega} \times \mathbf{u} - (\boldsymbol{\omega} \times \mathbf{r}) \times \text{curl } \mathbf{u} = -\omega \text{ grad } V + \nu \nabla^2 \mathbf{u} - (\mathbf{Q} \cdot \mathbf{r}) \boldsymbol{\omega} , \quad (2.7)$$

where

$$\omega V = W + \frac{p}{\rho} + \frac{1}{2} \mathbf{u}^2 - (\boldsymbol{\omega} \times \mathbf{r})^2 + 2(\boldsymbol{\omega} \cdot \mathbf{r})(\mathbf{Q} \cdot \mathbf{r}) - \frac{5}{2}(\mathbf{Q} \cdot \boldsymbol{\omega})r^2 .$$

The first term in (2.7) in brackets was omitted by Bondi and Lyttleton because they assumed that the motion is steady. So far as the motion of the fluid is concerned, the only significant difference between (2.5) and (2.7) is the factor 2 in the last term of (2.5). We shall show (§3) that this factor is crucial for obtaining a consistent solution of the problem when  $S_0$  is a sphere. The equation of continuity (2.4) is equivalent to

$$\text{div } \mathbf{u} = 0 . \quad (2.8)$$

Now choose a system of cylindrical polar coordinates  $(r, \theta, z)$  in  $\mathcal{F}$  with  $L_B$  as the  $z$ -axis,  $r$  measuring the distance of a typical point  $P(\mathbf{r})$  from  $L_B$ , and  $\theta$  being the angle between the plane defined by  $P$  and  $L_B$  and the plane defined by  $L_S$  and  $L_B$ . Then

$$(\mathbf{Q} \cdot \mathbf{r}) \boldsymbol{\omega} = (0, 0, -\Omega \omega z \cos \alpha - \Omega \omega r \sin \alpha \cos \theta) , \quad (2.9)$$

where  $\Omega = -|\mathbf{Q}|$  is the (retrograde) angular velocity of precession.

To begin with, neglect viscosity, and denote the velocity by  $u_0$  with components  $(u_{0r}, u_{0\theta}, u_{0z})$  respectively along the directions of  $r, \theta$  and  $z$  increasing; (2.5) and (2.8) are then equivalent to

$$\frac{\partial u_{0r}}{\partial t} + \omega \frac{\partial u_{0r}}{\partial \theta} - 2\omega u_{0\theta} = -\omega \frac{\partial V''}{\partial r} ,$$

$$\frac{\partial u_{0\theta}}{\partial t} + \omega \frac{\partial u_{0\theta}}{\partial \theta} + 2\omega u_{0r} = -\frac{\omega}{r} \frac{\partial V''}{\partial \theta} ,$$

(2.10)

$$\frac{\partial u_{0z}}{\partial t} + \omega \frac{\partial u_{0z}}{\partial \theta} = -\omega \frac{\partial V''}{\partial z} + 2\omega \Omega r \sin \alpha \cos \theta ,$$

$$\frac{1}{r} \frac{\partial}{\partial r} (ru_{0r}) + \frac{1}{r} \frac{\partial u_{0\theta}}{\partial \theta} + \frac{\partial u_{0z}}{\partial z} = 0 ,$$

where

$$V'' = V' - \Omega z^2 \cos \alpha - ru_{0\theta} .$$

The boundary conditions require that the normal component of the fluid velocity is zero on  $S_0$ , and also that, at  $t = 0+$ ,  $u + \Omega L$  is irrotational with velocity potential  $\phi_1$ , (say).

In order to solve (2.10) it is convenient to take the Laplace transform of the dependent variables with respect to time, denoting the result by a star, e.g.,

$$u_{0r}^*(r, \theta, z, s) = \int_0^\infty e^{-\omega s t} u_{0r}(r, \theta, z, t) dt. \quad (2.11)$$

The governing equations then reduce to

$$\begin{aligned} \left( \frac{\partial}{\partial \theta} + s \right) u_{0r}^* - 2u_{0\theta}^* &= -\frac{2V''^*}{\partial r} + \frac{1}{\omega} (u_{0r})_1, \\ \left( \frac{\partial}{\partial \theta} + s \right) u_{0\theta}^* + 2u_{0r}^* &= -\frac{1}{r} \frac{\partial V''^*}{\partial \theta} + \frac{1}{\omega} (u_{0\theta})_1, \\ \left( \frac{\partial}{\partial \theta} + s \right) u_{0z}^* &= -\frac{\partial V''^*}{\partial z} + \frac{1}{\omega} (u_{0z})_1 + \frac{2\Omega \sin \alpha}{\omega s} r \cos \theta, \end{aligned} \quad (2.12)$$

where the suffix 1 denotes initial conditions. Since, at  $t = 0+$ ,  $\underline{u} + \underline{\Omega} \times \underline{r}$  is derivable from the potential  $\phi_1$ , equations (2.12) may be simplified, on writing

$$V^* = V''^* - \frac{1}{\omega} \phi_1 + \frac{\Omega}{\omega} r z \sin \alpha \sin \theta \quad (2.13)$$

to



$$\left(\frac{\partial}{\partial \theta} + s\right) u_{0r}^* - 2u_{0\theta}^* = -\frac{\partial V^*}{\partial r} ,$$

$$\left(\frac{\partial}{\partial \theta} + s\right) u_{0\theta}^* + 2u_{0r}^* = -\frac{1}{r} \frac{\partial V^*}{\partial \theta} + \frac{\Omega}{\omega} r \cos \alpha , \quad (2.14)$$

$$\left(\frac{\partial}{\partial \theta} + s\right) u_{0z}^* = -\frac{\partial V^*}{\partial z} + \frac{2\Omega}{\omega s} r \sin \alpha (\cos \theta + s \sin \theta) ,$$

and the equation of continuity to

$$\frac{1}{r} \frac{\partial}{\partial r} (ru_{0r}^*) + \frac{1}{r} \frac{\partial u_{0\theta}^*}{\partial \theta} + \frac{\partial u_{0z}^*}{\partial z} = 0 . \quad (2.15)$$

A further simplification is obtained by writing

$$u_{0r}^* = \mathcal{R}(\bar{u}_{0z} e^{i\theta}) , \quad u_{0\theta}^* = \frac{\Omega r \cos \alpha}{\omega s} + \mathcal{R}(\bar{u}_{0\theta} e^{i\theta}) , \quad (2.16)$$

$$u_{0z}^* = \mathcal{R}(\bar{u}_{0z} e^{i\theta}) , \quad V^* = \frac{\Omega r^2 \cos \alpha}{\omega s} + \mathcal{R}(\bar{V} e^{i\theta}) ,$$

where  $\bar{u}_{0r}$ ,  $\bar{u}_{0\theta}$ ,  $\bar{u}_{0z}$ , and  $\bar{V}$  are independent of  $\theta$ . On substituting into (2.14) and (2.15), we obtain

$$\begin{aligned}\bar{u}_{0r} &= -\frac{1}{(s+1)^2 + 4} \left[ (s+1) \frac{\partial \bar{V}}{\partial r} + \frac{2i\bar{V}}{r} \right] , \\ \bar{u}_{0\theta} &= -\frac{1}{(s+1)^2 + 4} \left[ 2i \frac{\partial \bar{V}}{\partial r} + (s+1) \frac{\bar{V}}{r} \right] ,\end{aligned}\quad (2.17)$$

$$\bar{u}_{0z} = -\frac{2i\Omega \sin \alpha}{\omega s} r - \frac{1}{(s+1)} \frac{\partial \bar{V}}{\partial z} ,$$

where  $\bar{V}$  satisfies

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \bar{V}}{\partial r} \right) - \frac{\bar{V}}{r^2} + \frac{[(s+1)^2 + 4]}{(s+1)^2} \frac{\partial^2 \bar{V}}{\partial z^2} = 0 . \quad (2.18)$$

If  $S_0$  is given by  $f(r, z) = 1$ , the boundary condition to be satisfied is

$$\bar{u}_{0r} \frac{\partial f}{\partial r} + \bar{u}_{0z} \frac{\partial f}{\partial z} = 0 \quad \text{at } f = 1 . \quad (2.19)$$

### 3. THE SECONDARY INVISCID FLOW FOR A SPHEROID

The problem posed by equations (2.17) to (2.19) can be reduced to a solution of Laplace's equation on writing  $[(s+1)^2 + 4]/(s+1)^2 = \xi^2$ , and replacing  $z$  by  $\xi z$ . We can expect, therefore, that it can always be solved for the boundary condition (2.19), which is equivalent to a linear relation between  $\bar{V}$ , its normal derivative, and possibly also its tangential derivative. Having obtained the solution, the ultimate flow may be found by letting  $s \rightarrow 0$ , and taking note of the poles and branch points in the solution in the half plane  $\Re(s) \geq 0$ .

In this paper we are particularly concerned with the properties of the solution when the envelope  $S_0$  is the oblate spheroid

$$\frac{r^2}{a^2} + \frac{z^2}{b^2} = 1, \quad (a \geq b), \quad (3.1)$$

of which Bondi and Lyttleton considered the special case  $a = b$ . For the boundary (3.1), (2.19) becomes

$$\frac{r}{a^2} \bar{u}_{0r} + \frac{z}{b^2} \bar{u}_{0z} = 0, \quad (3.2)$$

and the appropriate solution of (2.18) is found by writing

$$\bar{V} = Arz, \quad (3.3)$$

where  $A$  is independent of  $r$  and  $z$ . Substituting into (2.17) we have

$$\bar{u}_{0r} = -\frac{Az}{(s-1)}, \quad \bar{u}_{0\theta} = -\frac{iAz}{(s-1)}, \quad \bar{u}_{0z} = -\frac{iA}{(s+1)} - \frac{2i\Omega r \sin \alpha}{\omega s} \quad (3.4)$$

Hence, from (3.2),

$$A = -\frac{2i\Omega a^2 \sin \alpha (s^2 + 1)}{\omega s [a^2 (s-1) + b^2 (s+1)]} \quad (3.5)$$

Inverting the solution with respect to  $s$ , and using (2.16), we have

$$u_{0r} = \frac{2\Omega za^2 \sin \alpha}{(a^2 - b^2)} \left[ \sin \theta - \frac{2a^2}{(a^2 + b^2)} \sin(\theta + \omega kt) \right], \quad (3.6)$$

$$u_{0\theta} = \frac{2\Omega za^2 \sin \alpha}{(a^2 - b^2)} \left[ \cos \theta - \frac{2a^2}{(a^2 + b^2)} \cos(\theta + \omega kt) \right] + \Omega r \cos \alpha \quad (3.7)$$

$$u_{0z} = -\frac{2\Omega rb^2 \sin \alpha}{(a^2 - b^2)} \left[ \sin \theta - \frac{2a^2}{(a^2 + b^2)} \sin(\theta + \omega kt) \right], \quad (3.8)$$

where

$$k = \frac{a^2 - b^2}{a^2 + b^2}.$$

According to (3.6) to (3.8), the secondary motion of the fluid consists of two parts: one is essentially due to the initial motion of the boundaries; the other is steady. The first part can itself be divided into

two components. One is rigid body rotation of angular velocity  $\Omega \cos \alpha$  about  $L_B$  which the rotation of the boundary failed to communicate to the fluid. The other is a free oscillation of the fluid. The steady part of the motion consists of the rigid body rotation  $\omega$  (cf. eq. [2.1]), and a circulation (given by the terms in  $\sin \theta$ ,  $\cos \theta$  and  $\sin \theta$  in [3.6], [3.7] and [3.8], respectively) in planes perpendicular to  $L_S$  the streamlines being similar and similarly situated ellipses. The vorticity of this circulation is constant, and equal to

$$- 2\Omega \sin \alpha \frac{(a^2 + b^2)}{(a^2 - b^2)} .$$

In the case of a sphere  $a = b$ , the relevant periods of free oscillation of the fluid are infinite, so that a resonance develops and

$$\left. \begin{aligned} u_{0r} &= -\Omega z \sin \alpha (\sin \theta + \omega t \cos \theta), \\ u_{0\theta} &= \Omega z \sin \alpha (\cos \theta + \omega t \sin \alpha) + \Omega r \cos \alpha, \\ u_{0z} &= \Omega r \sin \alpha (\sin \theta + \omega t \cos \theta), \end{aligned} \right\} \quad (3.9)$$

or, in vector notation,

$$\mathcal{U} = -\mathcal{L} \times \mathcal{L} - t(\mathcal{L} \times \omega) \times \mathcal{L} . \quad (3.10)$$

The physical interpretation of this result is that, as one expects, the motion of a spherical boundary exerts no influence on the motion of the fluid and hence, relative to the frame  $\mathcal{F}$ , the axis of rotation of the fluid rotates with angular velocity  $-\Omega$  about  $L_g$ . Consequently, after time  $t$  ( $\Omega t \ll 1$ ), the fluid rotates about an axis making an angle of

$$\left| \frac{\Omega t \times \underline{u}}{|\underline{u}|} \right|$$

with  $L_g$ , and this is indicated by (3.10). It is noted that, on substituting (3.10) into (2.7), the equations of motion are satisfied identically, and so any change in the motion of the fluid must be initiated through a boundary layer arising from the non-satisfaction of the requirement that  $\underline{u} = 0$  on the boundary  $S_0$ . Had the equation (2.7) required by Bondi and Lyttleton been used,  $t$  would be replaced by  $\frac{1}{2}t$  in (3.10), and the new formula for would not be consistent with this physical argument.

#### 4. THE BOUNDARY LAYER

The inviscid solution obtained in the previous section satisfies not only the inviscid equations but also the viscous equations (2.5). Viscosity therefore manifests itself only through the fact that this solution does not satisfy the viscous boundary condition  $u = 0$  on the spheroidal envelope  $S_0$ , and leads to a boundary layer whose thickness we can anticipate to be  $O(v^{\frac{1}{2}})$  if  $v$  is small where the adjustments in the tangential components of velocity are made. We shall calculate this boundary layer in the present section and its effect on the inviscid flow in the interior will be discussed in the following sections.

Let us write

$$u = u_0 + u_1, \quad (4.1)$$

where  $u$  is defined by (2.1),  $u_0$  is its value if  $v = 0$ , and  $u_1$  is the correction due to viscosity. Then, to order  $v^{\frac{1}{2}}$ ,  $u_1$  is the contribution from the boundary layer to  $u$ . Introduce spheroidal polar coordinates  $(\lambda_1, \mu_1, \theta)$ , where  $\lambda_1$  and  $\mu_1$  are defined by

$$r = (\lambda_1^2 + c^2)^{\frac{1}{2}}(1 - \mu_1^2)^{\frac{1}{2}}, \quad z = \lambda_1 \mu_1, \quad (4.2)$$

$c^2 = a^2 - b^2$  and the spheroid  $S_0$  is given by  $\lambda_1 = b$ . Then, using (3.6) to

(3.8), we see that the boundary condition  $\mu = 0$  requires that, at

$$\lambda_1 = b,$$

$$u_{1\lambda_1} = 0,$$

$$u_{1\mu_1} = \frac{2\Omega ab \sin \alpha}{(a^2 - b^2)} \left[ \sin \theta - \frac{2a^2}{(a^2 + b^2)} \sin(\theta + \omega t) \right] (b^2 + \mu_1^2 c^2)^{\frac{1}{2}}, \quad (4.3)$$

$$u_{1\theta} = -\frac{2\Omega a^2 b \sin \alpha}{(a^2 - b^2)} \left[ \cos \theta - \frac{2a^2}{(a^2 + b^2)} \cos(\theta + \omega t) \right] \mu_1 - \Omega a \cos \alpha (1 - \mu_1^2)^{\frac{1}{2}}$$

Beyond the boundary layer, when  $(b - \lambda_1)\nu^{\frac{1}{2}}$  is large and positive,  $\mu_1 = o(1)$ ; we shall then be particularly interested in  $u_{1\lambda_1}$  which is  $O(\nu^{\frac{1}{2}})$  and, unlike the other components of velocity, is not exponentially small. Consequently it engenders a tertiary flow through the interior of  $S_0$ . From the boundary condition (4.3) the flow in the boundary layer may be divided into two parts one of which ( $\mu_2$ ) is dependent on  $\theta$ , and one which ( $\mu_3$ ) is independent of  $\theta$ . We consider the part dependent on  $\theta$  first.

Write the Laplace transform of  $\mu_2$  with respect to  $\omega t$  as

$$\mathcal{R}(\bar{\mu}_2 e^{i\theta}), \quad (4.4)$$

and in the boundary layer make the conventional assumption that the operator



$\partial/\partial\lambda_1 = O(\nu^{\frac{1}{2}})$  while  $\partial/\partial\mu_1$  and  $\partial/\partial\theta$  are  $O(1)$ . Then as is usual with Ekman flows, in the governing equations  $\theta$  and  $\mu_1$  may be taken constant.  $\lambda_1$  may be replaced by  $b$  except when it appears in  $\partial/\partial\lambda_1$ . The governing equations reduce to

$$(s+1)\bar{u}_{2\mu_1} + \frac{2\mu_1 a}{(b^2 + c^2 \mu_1^2)^{\frac{1}{2}}} \bar{u}_{2\theta} = \frac{a^2 \nu}{\omega(b^2 + c^2 \mu_1^2)} \frac{\partial^2 \bar{u}_{2\mu_1}}{\partial \lambda_1^2}, \quad (4.5)$$

$$(s+1)\bar{u}_{2\theta} - \frac{2\mu_1 a}{(b^2 + c^2 \mu_1^2)^{\frac{1}{2}}} \bar{u}_{2\mu_1} = \frac{a^2 \nu}{\omega(b^2 + c^2 \mu_1^2)} \frac{\partial^2 \bar{u}_{2\theta}}{\partial \lambda_1^2}, \quad (4.6)$$

$$a(b^2 + c^2 \mu_1^2)^{\frac{1}{2}} \frac{\partial \bar{u}_{2\lambda_1}}{\partial \lambda_1} + \frac{\partial}{\partial \mu_1} [(1 - \mu_1^2)^{\frac{1}{2}} (b^2 + c^2 \mu_1^2)^{\frac{1}{2}} \bar{u}_{2\mu_1}] + \frac{1 \bar{u}_{2\theta} (b^2 + c^2 \mu_1^2)}{a(1 - \mu_1^2)^{\frac{1}{2}}} = 0 \quad (4.7)$$

and the boundary conditions to

$$\bar{u}_{2\lambda_1} = 0, \quad \bar{u}_{2\mu_1} = \frac{2i\Omega ab \sin \alpha (b^2 + c^2 \mu_1^2)^{\frac{1}{2}} (s+1)}{\omega s[s(a^2 + b^2) - 1(a^2 - b^2)]}, \quad \bar{u}_{2\theta} = \frac{2\Omega a^2 b \sin \alpha (s+1)\mu_1}{\omega s[s(a^2 + b^2) - 1(a^2 - b^2)]}$$

at  $\lambda_1 = b$ . Also  $\bar{u}_{2\mu_1}$  and  $\bar{u}_{2\theta} \rightarrow 0$  as  $(b - \lambda_1)^{\frac{1}{2}}$

Integrating (4.5) (4.6) we have

$$\begin{aligned}
 u_{2\mu_1} &= \frac{\Omega ab i \sin \alpha(s+1)}{\omega s[s(a^2+b^2)-1(a^2-b^2)]} \left\{ [a\mu_1 + \sqrt{(b^2+c^2\mu_1^2)}] e^{\delta_2(\lambda_1-b)} - \right. \\
 &\quad \left. - [a\mu_2 - \sqrt{(b^2+c^2\mu_1^2)}] e^{\delta_1(\lambda_1-b)} \right\} \\
 u_{2\theta} &= \frac{\Omega ab \sin \alpha(s+1)}{\omega s[s(c^2+b^2)-1(a^2-b^2)]} \left\{ [a\mu_1 + \sqrt{(b^2+c^2\mu_1^2)}] e^{\delta_2(\lambda_1-b)} + \right. \\
 &\quad \left. + [a\mu_1 - \sqrt{(b^2+c^2\mu_1^2)}] e^{\delta_1(\lambda_1-b)} \right\}
 \end{aligned} \tag{4.8}$$

where

$$\delta_2^2 = \frac{\omega}{va^2} (b^2 + c^2\mu_1^2) \left[ s+1 - \frac{2i\mu_1 a}{\sqrt{(b^2+c^2\mu_1^2)}} \right], \tag{4.9}$$

$$\delta_1^2 = \frac{\omega}{va^2} (b^2 + c^2\mu_1^2) \left[ s+1 + \frac{2i\mu_1 a}{\sqrt{(b^2+c^2\mu_1^2)}} \right],$$

and the signs of the square root of the expression for  $\delta_1$  and  $\delta_2$  are decided by requiring that, when  $s$  is large, real and positive,  $\delta_1$  and  $\delta_2$  are both positive. The value of  $\mu_{2\lambda_1}$  now follows by the direct integration of (4.7) so that all properties of the boundary layer are now formally known. It is noted

that the assumptions on which this boundary layer theory is built are consistent provided only that  $\delta_1$  and  $\delta_2$  do not vanish. Exceptional cases arise only if  $s$  is purely imaginary and  $|s| < 3$ , when the assumptions break down on certain circles  $\mu_1 = \text{constant}$ . Conceivably this may be serious at large times since singularities occur in the  $s$ -plane on the imaginary axis; on inverting we could get contributions to  $\mu_2$  of order  $t^n e^{is_0 t}$ , ( $n > 0$ ). However, it is believed that this does not happen. In a closely related problem (9) the flow has been studied in the neighbourhood of these lines, and it has been shown that main effect is to change the thickness of the layer from  $O(\nu^{\frac{1}{2}})$  to  $O(\nu^{\frac{2}{3}})$ . Adapting the argument to the present problem similar results are obtained, and it is found that, in the neighbourhood of the singular lines,  $\mu_{2\lambda_1} = O(\nu^{\frac{2}{3}})$ , instead of  $O(\nu^{\frac{1}{2}})$  as elsewhere in the boundary layer. Further, on computing the contribution from the neighbourhood of these lines to the flow in the deep interior of the spheroid, we find it to be  $O(\nu^{\frac{2}{3}})$  as against  $O(\nu^{\frac{1}{2}})$  from the rest of the boundary layer.

Of particular interest is the value  $\bar{u}_{2\lambda_1}$  just outside the boundary layer i. e. when  $(b - \lambda_1)\nu^{\frac{1}{2}}$  is large, but  $(b - \lambda_1)$  is small. This is obtained by integrating (4.7) with respect to  $\lambda_1$  from  $-\infty$  to  $b$  and using (4.9). Denoting the value of  $\bar{u}_{2\lambda_1}$  by  $\bar{u}_{4\lambda_1}(b, \mu_1, \theta)$  we have

$$\begin{aligned}
 & a(b^2 + c^2 \mu_1^2)^{\frac{1}{2}} \bar{u}_{4\lambda_1}(b, \mu_1, \theta) = \\
 & = \frac{\Omega a b i \sin \alpha(s+1)}{\omega s[s(a^2 + b^2) - 1(a^2 - b^2)]} \left[ \frac{\partial}{\partial \mu_1} \left\{ (1 - \mu_1^2)^{\frac{1}{2}} (b^2 + c^2 \mu_1^2)^{\frac{1}{2}} \left[ \frac{a\mu_1 + \sqrt{(b^2 + c^2 \mu_1^2)}}{\delta_2} - \frac{a\mu_1 - \sqrt{(b^2 + c^2 \mu_1^2)}}{\delta_1} \right] \right\} \right. \\
 & \quad \left. + \frac{(b^2 + c^2 \mu_1^2)}{a(1 - \mu_1^2)^{\frac{1}{2}}} \left\{ \frac{a\mu_1 + (b^2 + c^2 \mu_1^2)^{\frac{1}{2}}}{\delta_2} + \frac{a\mu_1 - (b^2 + c^2 \mu_1^2)^{\frac{1}{2}}}{\delta_1} \right\} \right] . \quad (4.10)
 \end{aligned}$$

Now let us consider the second part ( $\mu_3$ ) of the boundary layer due to that part of (4.3) independent of  $\theta$ . Let the Laplace transform of  $\mu_3$  with respect to  $\omega t$  be  $\mu_3^*$ . Then  $\lambda_1 = b$

$$u_{3\lambda_1}^* = 0, \quad u_{3\mu_1}^* = 0, \quad u_{3\theta}^* = -\Omega a \cos \alpha(1 - \mu_1^2)^{\frac{1}{2}} / \omega s, \quad (4.11)$$

and by an analogous argument to that given above the equations governing the flow in the boundary layer reduce to

$$s u_{\mu_1}^* + \frac{2\mu_1 a}{(b^2 + c^2 \mu_1^2)^{\frac{1}{2}}} u_{3\theta}^* = \frac{a^2 \nu}{\omega(b^2 + c^2 \mu_1^2)} \frac{\partial^2 u_{3\mu_1}^*}{\partial \lambda_1^2}, \quad (4.12)$$

$$s u_{3\theta}^* - \frac{2\mu_1 a}{(b^2 + c^2 \mu_1^2)^{\frac{1}{2}}} u_{3\mu_1}^* = \frac{a^2 \nu}{\omega(b^2 + c^2 \mu_1^2)} \frac{\partial^2 u_{3\theta}^*}{\partial \lambda_1^2}, \quad (4.13)$$

$$a(b^2 + c^2 \mu_1^2)^{\frac{1}{2}} \frac{\partial u_{3\lambda_1}^*}{\partial \lambda_1} + \frac{\partial}{\partial \mu_1} \left[ (1 - \mu_1^2)^{\frac{1}{2}} (b^2 + c^2 \mu_1^2)^{\frac{1}{2}} u_{3\mu}^* \right] = 0 \quad (4.14)$$

Integrating (4.12) and (4.13) we have

$$\left. \begin{aligned} u_{3\mu_1}^* &= - \frac{\Omega a l \cos \alpha (1 - \mu_1^2)^{\frac{1}{2}}}{2\omega s} \left[ e^{\delta_3(\lambda_1 - b)} - e^{\delta_4(\lambda_1 - b)} \right] , \\ u_{3\theta}^* &= - \frac{\Omega a \cos \alpha (1 - \mu_1^2)^{\frac{1}{2}}}{2\omega s} \left[ e^{\delta_3(\lambda_1 - b)} + e^{\delta_4(\lambda_1 - b)} \right] , \end{aligned} \right\} \quad (4.15)$$

where

$$\delta_3^2 = \frac{\omega}{\nu a} (b^2 + c^2 \mu_1^2) \left( s - \frac{2l\mu_1 a}{(b^2 + c^2 \mu_1^2)^{\frac{1}{2}}} \right) , \quad (4.16)$$

$$\delta_4^2 = \frac{\omega}{\nu a} (b^2 + c^2 \mu_1^2) \left( s + \frac{2l\mu_1 a}{(b^2 + c^2 \mu_1^2)^{\frac{1}{2}}} \right) ,$$

and  $\delta_3$  and  $\delta_4$  are real and positive when  $s$  is real, positive and large.

The consistency of this solution is subject to the same qualifications as

(4.10). Of particular interest here too is the value of  $u_{3\lambda_1}^*$  just outside the boundary layer, and denoting it by  $u_{3\lambda_1}^*(b, \mu_1, \theta)$  we have

$$a(b^2 + c^2 \mu_1^2)^{\frac{1}{2}} u_{5\lambda_1}^* = - \frac{\Omega a i \cos \alpha}{2\omega s} \frac{\partial}{\partial \mu_1} \left[ (b^2 + c^2 \mu_1^2)^{\frac{1}{2}} (1 - \mu_1^2)^{\frac{1}{2}} \left( \frac{1}{\delta_3} - \frac{1}{\delta_4} \right) \right]. \quad (4.17)$$

## 5. THE TERTIARY INVISCID FLOW

The boundary layer solution obtained in §4 shows that there must be a normal velocity at the edge of the layer (i. e. as  $\lambda_1 \rightarrow b^-$  on the inviscid scale) of order  $v^{\frac{1}{2}}$  and given by the sum of (4.10) and (4.17). In turn this must induce a tertiary inviscid flow throughout the interior of the spheroid, i. e. a flow governed by (2.14) and (2.15) excluding the forcing terms, and satisfying (2.19) with a right hand side effectively equal to the sum of (4.10) and (4.17) (instead of being zero). Since (4.10) is dependent on  $\theta$  while (4.17) is not, it is convenient to treat their contributions separately and in this section we shall consider the consequences of (4.10). In §6 below we shall consider the consequences of (4.17).

Denoting the corresponding velocity in the spheroid by  $u_4$  we have, from (2.17), that  $u_4$  can be expressed in terms of a scalar  $\bar{V}_4$  which satisfies (2.18), the forcing term in (2.17) being again set equal to zero. Hence, after stretching the  $z$  coordinate,  $\bar{V}_4$  satisfies Laplace's equation and the appropriate solution can be formally written down. Let

$$r = \frac{2}{[(s+1)^2 + 4]^{\frac{1}{2}}} (1 - \mu_4^2)^{\frac{1}{2}} (\lambda_4^2 + \gamma_4^2)^{\frac{1}{2}}, \quad z = \frac{2}{(s+1)} \mu_4 \lambda_4, \quad (5.1)$$

where

$$\gamma_4^2 = a^2 + \frac{1}{4} c^2 (s+1)^2. \quad (5.2)$$

In terms of  $\mu_4$  and  $\lambda_4$  the spheroid is given by

$$\lambda_4 = \frac{1}{2}(s+1)b \quad (5.3)$$

and on the spheroid  $\mu_4 = \mu_1 = \mu$ . The most general acceptable form for  $\bar{V}_4$  is

$$\bar{V}_4 = (1 - \mu_4^2)^{\frac{1}{2}} (\lambda_4^2 + \gamma_4^2)^{\frac{1}{2}} \sum_{n=1}^{\infty} A_n P_n' \left( \frac{i\lambda_4}{\gamma_4} \right) P_n'(\mu_4) , \quad (5.4)$$

the  $P_n$  being Legendre polynomials and the  $A_n$  constants to be found.

Further, the boundary condition associated with (4.10), viz.

$$\frac{r}{a} u_{4r} + \frac{z}{b} u_{4z} = \left( \frac{r^2}{a^2} + \frac{z^2}{b^2} \right)^{\frac{1}{2}} u_{4\lambda_1}(b, \mu, \theta) ,$$

becomes, in terms of  $\bar{V}_4$ ,

$$\frac{a}{2} \frac{\partial \bar{V}_4}{\partial \lambda_4} + \frac{2ib}{a[(s+1)^2 + 4]} \bar{V}_4 = (b^2 + c^2 \mu^2)^{\frac{1}{2}} \bar{u}_{4\lambda_1} \quad (5.5)$$

at  $\lambda_4 = (s+1)b/2$ . It follows from (4.10) that  $\bar{u}_{4\lambda_1}$  is an odd function



of  $\mu$  and behaves like  $(1-\mu^2)^{\frac{1}{2}}$  near  $\mu = \pm 1$ . Consequently  $n$  is even in (5.4), and the various  $A_n$  are determined from the equation

$$A_n \left\{ \frac{a}{2} \frac{\partial}{\partial \lambda_4} [(\lambda_4^2 + \gamma_4^2)^{\frac{1}{2}} P'_n(\frac{i\lambda_4}{\gamma_4})] + \frac{2ib(\lambda_4^2 + \gamma_4^2)^{\frac{1}{2}}}{a[(s+1)^2 + 4]} P'_n(\frac{i\lambda_4}{\gamma_4}) \right\} = B_n, \quad (5.6)$$

with  $\lambda_4 = (s+1)b/2$  and

$$\frac{2n(n+1)}{2n+1} B_n = \int_{-1}^{+1} (b^2 + c^2 \mu^2)^{\frac{1}{2}} \bar{u}_{4\lambda_1} P'_n(\mu) (1-\mu^2)^{\frac{1}{2}} d\mu. \quad (5.7)$$

From this point on, the determination of  $V_4$  is in principle straightforward but involves the evaluation of complicated inversion and other integrals. However, since  $v$  is assumed to be small the effect of the tertiary flow may be neglected unless the  $A_n$  have poles, qua functions of  $s$ , in the half plane  $\mathcal{R}(s) > 0$ , or sufficiently strong singularities on the imaginary axis of  $s$ . Our purpose in the rest of this section is to give arguments to exclude such possibilities.

First we show that  $A_n$  has no singularities in the half plane

$\mathcal{R}(s) > 0$ . A singularity can arise if either the coefficient of  $A_n$  in (5.6) vanishes or if  $B_n$  is singular, or if (2.18) breaks down. Taking these possibilities in order, the coefficient of  $A_n$  vanishes if

$$(s-1)bP'_n(\psi) + 2i\gamma_4 n(n+1)P_n(\psi) = 0, \quad (5.8)$$

using the properties of Legendre polynomials and writing

$$\psi = \frac{1(s+1)b}{[4a^2 + c^2(s+1)^2]^{\frac{1}{2}}} \quad (5.9)$$

Writing  $s = 1\sigma$  (5.8) reduces to

$$F(\sigma) \equiv (1-\sigma)\psi P'_n(\psi) + (1+\sigma)n(n+1)P_n(\psi) = 0 \quad (5.10)$$

where

$$\psi = - \frac{(1+\sigma)b}{[4a^2 - c^2(1+\sigma)^2]^{\frac{1}{2}}} \quad (5.11)$$

we shall now show that the roots of (5.10) occur at real values of  $\sigma$ . We observe that physically this result is to be expected because it means that the periods of free oscillation of the fluid in the spheroidal envelope are real. In order to prove this result, we note that (5.10) is effectively a polynomial of degree  $n+1$  in  $\sigma$  and hence it is enough to prove that  $F(\sigma)$  has  $n+1$  zeros on the real axis of  $\sigma$ . One such zero is clearly at  $\sigma = -1$ , where  $\psi = 0$ . Further, between  $\sigma = -1$  and  $\sigma = +1$ , where  $\psi = 1$ ,  $P_n$  has  $n/2$  zeros and consequently, from the interlacing property of the zeros of  $P_n$  and  $P'_n$ ,  $F(\sigma)$  has  $\frac{1}{2}n - 1$  zeros between the first positive zero of  $P_n$ , qua function of  $\psi$ , and  $\sigma = 1$ . A similar

remark applies to  $-3 < \sigma < -1$ . Again if  $n/2$  is even,  $F(\sigma) > 0$  if  $\sigma + 1$  is small and positive, and is negative at the first zero of  $P_n$  for which  $\sigma > -1$ . Hence  $F(\sigma)$  vanishes in this range too. A similar argument applies if  $n/2$  is odd. Counting up we see that there are exactly  $n + 1$  zeros of  $F(\sigma)$  in  $-3 < \sigma < 1$  all simple, and leading to finite oscillations of the fluid except when they coincide with poles of  $B_n$ .

Now consider the singularities of  $B_n$ . There are two simple poles at  $s = 0$  and at  $s = ik$ , from (3.8) and (4.10). In addition  $\delta_1$  and  $\delta_2$  vanish within the range  $-3i < s < i$  for any acceptable value of  $\mu$ . In the evaluation of  $B_n$  we need the weighted integral of  $\bar{u}_{4\lambda_1}$ ; consequently the associated singularities of  $B_n$  only occur at  $s = -3i$  and  $s = i$ , and will be considered at the same time as the singularities of the governing equation (2.18). The poles of  $B_n$  at  $s = 0$  and  $s = ik$  can lead to motions which are apparently large as  $t \rightarrow \infty$  if one of the zeros of (5.8) occurs at the same value of  $s$ . The second possibility ( $s = ik$ ) is unlikely if  $n \neq 2$  since it implies that two free periods of oscillation of the fluid are equal but we have not been able to rule this out. The first possibility ( $s = 0$ ) is real, and there seems little doubt that for any  $n$  we can choose a value of  $a/b$  to satisfy (5.8) by  $s = 0$ ; for example, if  $n = 4$  and

$$6a = (\sqrt{39} + \sqrt{15})b,$$

the left-hand side of (5.8) vanishes at  $s = 0$ . Using the argument above for  $s = ik$ , ( $k = (a^2 - b^2)/(a^2 + b^2)$ ), we can say that if  $a = b$  it is most

unlikely that (5.8) is satisfied by  $s = 0$  for  $n \neq 2$ . In the case of particular interest therefore, when  $a \approx b$ , we conclude that the possibility of satisfying (5.8) at  $s = 0$  is not likely to be serious except if  $n = 2$ .

Let us consider the case  $n = 2$  in some detail. Here  $A_2$  has a double pole at  $s = ik$  because the coefficient of  $A_2$  vanishes and because  $B_2$  has a simple pole from (4.10). Using the formula for  $A$  in (3.5)

$$B_2 = \frac{5}{4} \int_{-1}^1 \mu (1 - \mu^2)^{\frac{1}{2}} (b^2 + c^2 \mu^2)^{\frac{1}{2}} \bar{u}_{4\lambda_1} d\mu, \quad (5.12)$$

$$= - \frac{5A(s-1)}{4b} \int_{-1}^1 d\mu \frac{(b^2 + c^2 \mu^2)^{\frac{1}{2}}}{a\delta_2} [a\mu + (b^2 + c^2 \mu^2)^{\frac{1}{2}}][\mu(b^2 + c^2 \mu^2)^{\frac{1}{2}} - a(1 - 2\mu^2)]. \quad (5.13)$$

The corresponding value of  $A_2$  is now easily worked out from (5.6) and, reverting to the coordinates  $(r, \theta, z)$ , the corresponding contribution to  $\bar{V}_4$  is

$$\frac{3rz(s^2 + 1)B_2}{a^2(s-1) + b^2(s+1)} \quad (5.14)$$

Let the coefficient of  $A$  in (5.13) be  $\chi$ . The implication of (5.4) is that the secondary motion described by (3.3) and (3.5) gives rise to a complicated tertiary motion of which the second harmonic is of the same form as (3.3) but with a double pole at  $s = ik$ . Although formally this means that the tertiary flow increases without limit as  $t \rightarrow \infty$  it is to be

expected that the double pole arises through an error of order  $\chi$  in the position of the simple pole of  $A$  in (3.5). To see this we note that the contribution from (5.14) to the tertiary flow is ultimately dominant and it gives rise via the boundary layer to a similar form to (5.14) in the quaternary flow but now with a triple pole at  $s = ik$ , and so on. Adding up all these contributions to  $\bar{V}$  which are proportional to  $rz$ , qua functions of  $r$  and  $z$ , their total is

$$\begin{aligned} \text{Arz} \sum_{m=0}^{\infty} \left[ \frac{3(s^2+1)\chi}{a^2(s-1) + b^2(s+1)} \right]^m \\ = - \frac{2\Omega a^2 \sin \alpha (s^2+1) rz}{\omega s [a^2(s-1) + b^2(s+1) - 3(s^2+1)\chi]} \end{aligned} \quad (5.15)$$

Thus the effect of the boundary layer is to shift the pole  $s = ik$  to

$$s = ik + \frac{3(1-k^2)\chi}{a^2 + b^2} + \dots, \quad k = \frac{a^2 - b^2}{a^2 + b^2},$$

the neglected terms being  $O(\nu^{\frac{1}{2}})$  and most probably  $O(\nu^{\frac{1}{3}})$  from (9).

Of particular interest is the position of the pole when

$$1 \gg \frac{a^2 - b^2}{a^2 + b^2} \gg \frac{\nu}{\omega a^2}, \quad (5.16)$$

the second condition being fundamental to the present theory. So far as  $\chi$  is concerned it is sufficient to set  $a = b$  when

$$\chi = -\frac{5a^2}{4} \int_{-1}^1 \frac{(1+\mu)^2(1-2\mu)d\mu}{\delta_2}, \quad (5.17)$$

where

$$\delta_2 = \frac{1}{a} \left( \frac{\omega a^2}{\nu} \right)^{\frac{1}{2}} (s+1-2i\mu)^{\frac{1}{2}}$$

and since  $s \approx 0$

$$\delta_2 = \begin{cases} \frac{1}{a} \left( \frac{\omega a^2}{\nu} \right)^{\frac{1}{2}} e^{i\pi/4} (1-2\mu)^{\frac{1}{2}}, & \text{if } \mu < \frac{1}{2}, \\ \frac{1}{a} \left( \frac{\omega a^2}{\nu} \right)^{\frac{1}{2}} e^{-i\pi/4} (2\mu-1)^{\frac{1}{2}}, & \text{if } \mu > \frac{1}{2}. \end{cases} \quad (5.13)$$

It follows that

$$\chi = \frac{5i}{4} a^2 \left( \frac{\nu}{2\omega a^2} \right)^{\frac{1}{2}} (0.195 + 1.976i),$$

whence the pole is moved to

$$s = 1 \frac{(a^2 - b^2)}{(a^2 + b^2)} - \left( \frac{\nu}{2\omega a} \right)^{\frac{1}{2}} (3.720 - 0.3651) + \dots \quad (5.19)$$

and, since it is in the half plane  $\Re(s) < 0$ , the associated contribution to the flow dies out as  $t \rightarrow \infty$ . A similar result can be expected for all  $b/a$ .

The last cases to be discussed are the behaviour of the solution in the neighbourhood of  $s = \pm 1$  and  $-3i$ , where the differential equation (2.18) takes on a singular form. Although  $\delta_2$  vanishes at  $\mu a = (b^2 + c^2 \mu^2)^{\frac{1}{2}}$  when  $s = 1$  it follows from (4.10) and (5.7) that  $B_n$  is bounded as  $s \rightarrow 1$  for all  $n$ . Hence, substituting into (5.6) and noting that  $\lambda_4 = i\gamma_4$  on  $S_0$  if  $s = 1$ , it follows that

$$A_n = O(s - 1)^{\frac{1}{2}}, \quad (5.20)$$

near  $s = 1$ . Further the  $n^{\text{th}}$  harmonic in (5.4) can be written as

$$\frac{1}{4} [(s+1)^2 + 4]^{\frac{1}{2}} (s+1) r z A_n G_{n/2} \left\{ \frac{1}{4} [(s+1)^2 + 4] r^2, \frac{1}{4} (s+1)^2 z^2 \right\}, \quad (5.21)$$

where  $G_{n/2}(\alpha, \beta)$  is a homogeneous polynomial of degree  $n/2$  in  $\alpha$  and  $\beta$  with constant coefficients. Hence the contribution to the tertiary flow is

$O(s-1)$  as  $s \rightarrow 1$ , and its inverse with respect to  $s$  tends to zero as  $t \rightarrow \infty$ . A similar remark applies to the neighbourhood of  $s = -3i$ . In the neighbourhood of  $s = -1$ ,  $B_n$  is  $O(s+1)$  from (4.10), and the coefficient of  $A_n$  in (5.6) is  $O(1)$  since  $n$  is even. Consequently (5.21) is  $O(s+1)^3$  near  $s = -1$  and its contribution to the tertiary flow tends to zero as  $t \rightarrow \infty$ .

Summarizing we can expect that, of all the components of the secondary flow which depend on  $\theta$ , only the steady component will remain as  $t \rightarrow \infty$ , the boundary layer serving to damp out the oscillatory terms.



## 6. THE TERTIARY INVISCID FLOW (11)

In this section we consider the inviscid flow engendered by (4.17) which is independent of  $\theta$ . We assume that the associated inviscid flow  $u_5$  is also independent of  $\theta$  whence, on taking the Laplace transform with respect to  $\omega t$ , the governing equations reduce to

$$su_{5r}^* - 2u_{5\theta}^* = -\frac{\partial V_5^*}{\partial r}, \quad su_{5\theta}^* + 2u_{5r}^* = 0, \quad (6.1)$$

$$su_{5z}^* = -\frac{\partial V_5^*}{\partial z}, \quad \frac{\partial}{\partial r}(ru_{5r}^*) + \frac{\partial}{\partial z}(zu_{5z}^*) = 0;$$

those equations also follow from (2.12) on assuming that  $u_5$  is independent of  $\theta$  and neglecting the initial and forcing terms. Further, on  $S_0$ ,

$$\frac{r}{a} u_{5r}^* + \frac{z}{b} u_{5z}^* = \left[ \frac{r^2}{a^4} + \frac{z^2}{b^4} \right]^{\frac{1}{2}} u_{5\lambda_1}^*(b, \mu), \quad (6.2)$$

from (4.17). In terms of  $V_5^*$ ,

$$u_{5r}^* = -\frac{s}{s^2 + 4} \frac{\partial V_5^*}{\partial r}, \quad u_{5z}^* = -\frac{1}{s} \frac{\partial V_5^*}{\partial z}, \quad (6.3)$$

so that

$$\frac{\partial}{\partial r} \left[ r \frac{\partial V_5^*}{\partial r} \right] + \frac{(s^2 + 4)}{s^2} \frac{\partial^2 V_5^*}{\partial z^2} = 0 \quad (6.4)$$

Introduce new coordinates  $\mu_5$ ,  $\lambda_5$  such that

$$r = \frac{2}{s^2 + 4} (\lambda_5^2 + \gamma_5^2)^{\frac{1}{2}} (1 - \mu_5^2)^{\frac{1}{2}}, \quad z = \frac{2}{5} \mu_5 \lambda_5, \quad (6.5)$$

where  $\gamma_5 = a^2 + \frac{1}{4} s^2 c^2$ ,  $S_0$  is given by  $\lambda_5 = \frac{1}{2} b_5$  and, on  $S_0$ ,

$$\mu_5 = \mu_4 = \mu_1 = \mu.$$

The boundary conditions now reduce to

$$-\frac{\partial V_5^*}{\partial \lambda_5} = \frac{1}{a} (b^2 + c^2 \mu^2)^{\frac{1}{2}} u_{5\lambda}^*, \quad (6.6)$$

on  $S_0$ . The most general appropriate solution for  $V_5^*$  in  $S_0$  is

$$V_5^* = \sum_{n=0}^{\infty} C_n P_n \left( \frac{i\lambda_5}{\gamma_5} \right) P_n(\mu_5), \quad (6.7)$$

where  $C_n$  are constants and, since  $\mu_{5\lambda}^*$  is an even function of  $\mu$ ,  $n$  also must be even. Hence

$$\frac{2}{(2n+1)} C_n P'_n\left(\frac{ib_5}{2\gamma_5}\right) = -\frac{\Omega \cos \alpha}{2\omega_8} \int_{-1}^{+1} P_n(\mu) \frac{\partial}{\partial \mu} [(b^2 + c^2 \mu^2)^{\frac{1}{2}} \left(\frac{1}{\delta_3} - \frac{1}{\delta_4}\right)] d\mu \quad (6.8)$$

The main interest here centers on the behaviour of the solution at large times, contributions to which arise from three sources. To begin with  $C_n$  will develop a simple pole whenever  $P'_n$  vanishes unless  $s = 0$ : the corresponding motion, according to (6.7), is a feeble oscillation and one can expect to show, using a similar argument to that which led to (5.19), that viscosity damps it out ultimately. Further the differential equation is singular at  $s = \pm 2i$  where  $ib_5/2\gamma_5 = \pm 1$ . Since  $P'_n(\pm 1) \neq 0$ , the leading terms from this cause must become zero, as  $t \rightarrow \infty$ . Finally, one must consider the case  $s = 0$ . When  $s$  is very small

$$C_n = -\frac{(2n+1) \Omega \cos \alpha a^3}{n(n+1) b \omega_8^2 P_n(0)} \left(\frac{\nu}{\omega a^2}\right)^{\frac{1}{2}} \int_{-1}^{+1} d\mu (1 - \mu^2) P'_n(\mu) \frac{(b^2 + c^2 \mu^2)^{\frac{1}{4}}}{(-2i\mu a)^{\frac{1}{2}}}, \quad (6.9)$$

so that, on inverting (6.7),  $V_5^*$  contains a term which is proportional to  $t$  when  $t$  is large. The implication is that the boundary layer exerts a decisive influence on the inviscid flow outside in this instance.

The reason for the difference between the solution in §5 near  $s = -1$  and the solution in §6 near  $s = 0$ , although both governing equations are similar and singular at these points is two-fold. First (4.10) contains a factor  $(s+1)$  while (4.17) contains a factor  $s^{-1}$

so that the behaviour of  $V_5^*$  near  $s = 0$  is bound to be more singular than the behaviour of  $V_4^*$  near  $s = -1$ . There is, however, a second deeper reason. Near  $s = 0$  the governing equation (6.1) reduces to

$$u_{5r}^* = 0, \quad \frac{\partial u_{5z}^*}{\partial z} = 0, \quad (6.10)$$

which means that  $u_{5z}^*$  is independent of  $z$  and therefore must be an even function of  $\mu$ . Consequently  $V_5^*$  is an odd function of  $\mu$  but the boundary condition (6.2) requires it to be an even function. This patent contradiction means that a serious breakdown in the solution must occur as  $t \rightarrow \infty$ . From studies of the steady state (e.g. Proudman (8)) we know that the boundary layer in such a case exerts a decisive influence on the flow outside, adjusting it until the condition that  $u_{5\lambda}$  is an odd function of  $\mu$  is satisfied, which is only possible if  $u_{5\lambda} \rightarrow 0$  as  $t \rightarrow \infty$ . Hence the initial angular velocity  $u_0 = \Omega r \cos \alpha$  is also damped out by the boundary layer.

On the other hand near  $s = -1$ , while the governing equations (2.17) and (2.18) also reduce in part to

$$\frac{\partial u_{4z}^*}{\partial z} = 0, \quad (6.11)$$

so that  $u_{4z}^*$  is an odd function of  $\mu$ , this condition is actually satisfied

by (4.10) so that no corresponding difficulties occur.

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